Visual angle on the unit sphere.

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Abstract
Let AOB be a triangle in the three dimensional Euclidean space $\boldsymbol{R}^3$. When we look at this triangle from various viewpoints, the angle $\angle AOB$ changes its appearance, and its 'visual size' is not constant. This visual angle is realized on the unit sphere $\boldsymbol{S}^2$ centered at O. In this paper, we will investigate the contours of visual size in the unit sphere. Several geometric properties and a stochastic property are presented.

1. Introduction

We see various angles in our daily lives. Each angle has a certain size, however, its appearance changes according to our viewpoint. We call the appearance visual angle. Note that visual angle in this paper is not the angle a viewed object subtends at the eye which is called the object's angular size. This visual angle is determined by the true size of the angle and the position of the observer. The motivation of this research is to derive information about the relation between the angle and the observer from the visual angle.

Let AOB be a fixed angle determined by three points O, A, and B in the three dimensional Euclidean space $\mathbf{R}^3$. When we look at this angle, its appearance changes according to our viewpoint. The *visual angle* of $\angle AOB$ from a viewpoint P is defined as follows:

Definition 1.1(Visual Angle and Visual Size) Let $\angle AOB$ be a fixed angle determined by three points O, A, and B in the three dimensional Euclidean space $\mathbf{R}^3$. For a viewpoint P, let us denote by

$$\angle_P AOB$$

the dihedral angle of the two faces OAP and OBP of the (possibly degenerate) tetrahedron $POAB$. This angle $\angle_P AOB$ is called the *visual angle* of $\angle AOB$ from the viewpoint P. Its size (measure) is called the *visual size* of $\angle AOB$ from P, and denoted by the same notation $\angle_P AOB$ as the visual angle without any confusion.

For an angle with fixed size, its visual size can vary from 0 to π in radians depending on the viewpoint. We may suppose that two points A and B lie on the unit sphere $\mathbf{S}^2$ centered at O (see, Figure 1.1). Then the spherical distance $\overline{AB}$ between A and B is equal to $\angle AOB$. (We denote the shortest geodesic connecting A and B, and its length by the same notation $\overline{AB}$.) In addition, let us assume that a viewpoint P is also on $\mathbf{S}^2$. Then, notice that $\angle_P AOB$ is equal to the interior angle $\angle P$ of the spherical triangle ΔAPB, because the tangent plane of $\mathbf{S}^2$ at P is orthogonal to the line OP.

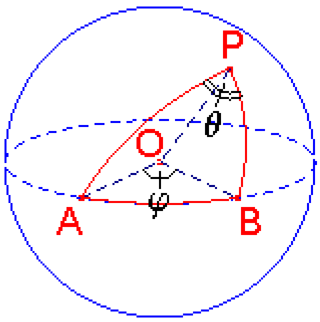

Figure 1.1 Visual angle on the unit sphere.

The aim of this paper is to study the following contours on $\mathbf{S}^2$:

$${}_{\varphi}C_{\theta} := \{\, P \in \mathbf{S}^2 \;\; | \; \angle AOB = \varphi,\; \angle_P AOB = \theta \,\},$$

where $0 < \varphi < \pi$ and $0 \le \theta \le \pi$ (In the cases that $\varphi = 0$ and $\varphi = \pi$, it is trivial that ${}_0C_0 = {}_{\pi}C_{\pi} = \mathbf{S}^2 \setminus \{A, B\}$). We can reduce the domain of φ to $(0, \pi/2]$. In fact,

$${}_{\varphi}C_{\theta} = \{\, P \in \mathbf{S}^2 \;\; | \; \angle A^*OB = \pi - \varphi,\; \angle_P A^*OB = \pi - \theta \,\},$$

where A^* is the antipodal point of A, therefore, ${}_{\varphi}C_{\theta}$ is essentially the same as ${}_{\pi-\varphi}C_{\pi-\theta}$. Hence, let us suppose that $0 < \varphi \le \pi/2$ in the following argument.

In Section 2, we will seek the algebraic expression of ${}_{\varphi}C_{\theta}$ and several geometric properties of ${}_{\varphi}C_{\theta}$. The stochastic property of ${}_{\varphi}C_{\theta}$ will be reviewed (see [4] and [5]) in Section 3. Figures in this paper are drawn by geometry software Cabri II and Maple.

2. Contour of visual angle on $\mathbf{S}^2$

In plane geometry, for two points A and B, the set of points P for which the angle $\angle APB$ equals $\pi/2$ is a circle with diameter AB. More generally, the set of points P for which the angle $\angle APB$ equals $\theta(\neq \pi/2)$ is the union of two arcs passing through two points A and B (Figure 2.1).

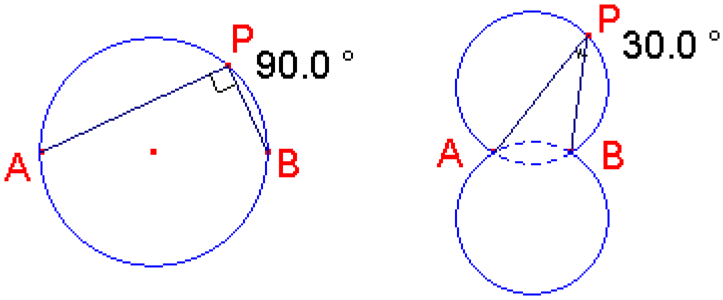

Figure 2.1 The locus of the point P such that $\angle APB$ is constant.

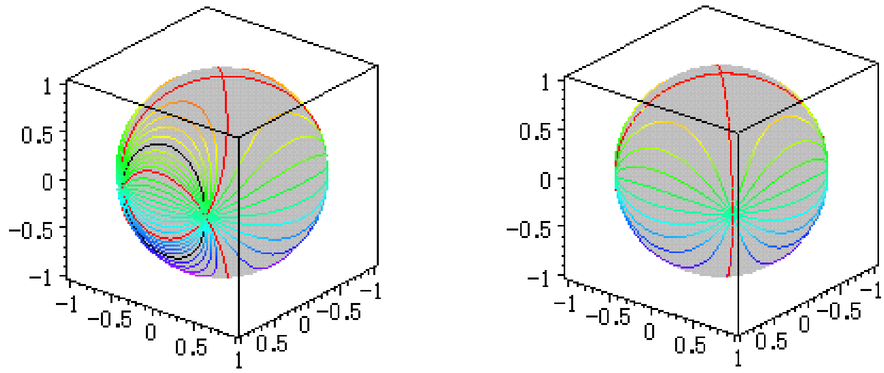

Figure 2.2 Contours on the unit sphere($\varphi = \pi/3$ (left), $\varphi = \pi/2$ (right)).

The contours ${}_{\varphi}C_{\pi/2}$ and ${}_{\varphi}C_{\theta}$ are regarded as a natural extension of the above geometric figures to spherical geometry, however, they are not so simple (see, Figure 2.2). Set the two points A and B on the equator of $\mathbf{S}^2$ (see, Figure 1.1) as follows:

Theorem 2.1 For two points $A = (\cos(\varphi/2), -\sin(\varphi/2), 0)$ and $B = (\cos(\varphi/2), \sin(\varphi/2), 0)$, the set ${}_{\varphi}C_{\theta}$ of points $P = (x, y, z)$ for which the angle $\angle APB$ equals θ ($\angle_P AOB = \theta$) on $\mathbf{S}^2$ satisfies the equations

$$\begin{cases} (2\sin\theta)y^2 + \sin\theta(1+\cos\varphi)z^2 - (2\cos\theta\sin\varphi)z = \sin\theta(1-\cos\varphi) & (z \geq 0), \\ (2\sin\theta)y^2 + \sin\theta(1+\cos\varphi)z^2 + (2\cos\theta\sin\varphi)z = \sin\theta(1-\cos\varphi) & (z \leq 0). \end{cases} \tag{2.1}$$

Proof. By the symmetry of $\mathbf{S}^2$, it is sufficient to prove the case that $z \geq 0$. Apply the law of cosines for sides to the spherical triangle ΔAPB ([1] p.54):

$$\cos\varphi = \cos a \cos b + \sin a \sin b \cos\theta \tag{2.2}$$

where $a = \overline{BP}$ and $b = \overline{AP}$.

Since $\cos a = x\cos\frac{\varphi}{2} + y\sin\frac{\varphi}{2}$ and $\cos b = x\cos\frac{\varphi}{2} - y\sin\frac{\varphi}{2}$,

$$\cos a \cos b = \tfrac{1}{2}\{(1+\cos\varphi)x^2 - (1-\cos\varphi)y^2\}, \tag{2.3}$$

$$\cos^2 + \cos^2 b = (1+\cos\varphi)x^2 + (1-\cos\varphi)y^2. \tag{2.4}$$

Using these equations above, let us show the equation $\sin a \sin b \sin\theta = z\sin\varphi$.

$$\begin{aligned} \sin^2 a \sin^2 b \sin^2\theta &= \sin^2 a \sin^2 b(1-\cos^2\theta) \\ &= (1-\cos^2 a)(1-\cos^2 b) - (\cos\varphi - \cos a\cos b)^2 \quad (\textit{by Equation } (2.2)) \\ &= \sin^2\varphi - (1+\cos\varphi)x^2 - (1-\cos\varphi)y^2 + \{(1+\cos\varphi)x^2 - (1-\cos\varphi)y^2\}\cos\varphi \\ &\qquad\qquad (\textit{by Equations } (2.3) \textit{ and } (2.4)) \\ &= \sin^2\varphi - (x^2+y^2)\sin^2\varphi \\ &= z^2\sin^2\varphi. \end{aligned}$$

Since $\sin a, \sin b, \sin\theta, \sin\varphi, z \geq 0$, one has $\sin a \sin b \sin\theta = z\sin\varphi$.

If $\theta \neq \pi/2$,

$$\begin{aligned} 2z\sin\varphi\cos\theta &= 2\sin a\sin b\sin\theta\cos\theta \\ &= 2\sin\theta(\cos\varphi - \cos a\cos b) && (\textit{by Equation } (2.2)) \\ &= \sin\theta\{2\cos\varphi - ((1+\cos\varphi)x^2 - (1-\cos\varphi)y^2)\} && (\textit{by Equation } (2.3)) \\ &= \sin\theta\{2\cos\varphi - ((1+\cos\varphi)(1-y^2-z^2) - (1-\cos\varphi)y^2)\} \\ &= \sin\theta\{2y^2 + (1+\cos\varphi)z^2 + \cos\varphi - 1\}, \end{aligned}$$

hence,

$$(2\sin\theta)y^2 + \sin\theta(1+\cos\varphi)z^2 - (2\cos\theta\sin\varphi)z = \sin\theta(1-\cos\varphi). \tag{2.5}$$

In the case that $\theta = \pi/2$, $\cos\varphi = \cos a\cos b$ by Equation (2.2). Combining this equation with Equation (2.3) one has

$$2y^2 + (1+\cos\varphi)z^2 = 1-\cos\varphi. \tag{2.6}$$

This equation is a special case of Equation (2.5) with $\theta = \pi/2$. This completes the proof. ■

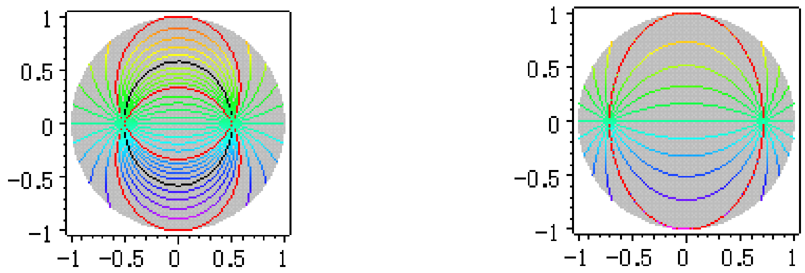

Figure 2.3 View of contours from x-axis($\varphi = \pi/3$ (left), $\varphi = \pi/2$ (right)).

Figure 2.3 shows the views of contours from x-axis. Theorem 2.1 indicates that ${}_{\varphi}C_{\theta}$ is given as a part of intersection curves of elliptic cylinders and $\mathbf{S}^2$.

Now, let us investigate geometric properties of ${}_{\varphi}C_{\theta}$. First, let us focus on the relation between ${}_{\varphi}C_{\theta}$ and ${}_{\varphi}C_{\pi-\theta}$. In plane geometry, for two points A and B, the set of points P for which the angle $\angle APB$ equals $\pi/6$ and the set of points P for which the angle $\angle APB$ equals $\pi - \pi/6$ make two circles as in Figure 2.1(right). A similar feature is shown in ${}_{\varphi}C_{\theta}$ and ${}_{\varphi}C_{\pi-\theta}$.

Proposition 2.1 For $\varphi \neq \pi/2$, two sets ${}_{\varphi}C_{\theta}$ and ${}_{\varphi}C_{\pi-\theta}$ are complementary to each other, i.e., the set ${}_{\varphi}C_{\theta} \cup {}_{\varphi}C_{\pi-\theta}$ satisfies

$$(2\sin\theta)y^2 + \sin\theta(1+\cos\varphi)z^2 \pm (2\cos\theta\sin\varphi)z = \sin\theta(1-\cos\varphi).$$

Proof. By Equations (2.1), the set ${}_{\varphi}C_{\pi-\theta}$ satisfies

$$\begin{cases} (2\sin\theta)y^2 + \sin\theta(1+\cos\varphi)z^2 + (2\cos\theta\sin\varphi)z = \sin\theta(1-\cos\varphi) & (z \geq 0), \\ (2\sin\theta)y^2 + \sin\theta(1+\cos\varphi)z^2 - (2\cos\theta\sin\varphi)z = \sin\theta(1-\cos\varphi) & (z \leq 0). \end{cases}$$

Hence the intersection curves of $\mathbf{S}^2$ and the elliptic cylinder defined by

$$(2\sin\theta)y^2 + \sin\theta(1+\cos\varphi)z^2 - (2\cos\theta\sin\varphi)z = \sin\theta(1-\cos\varphi)$$

are divided into two parts, that is, the upper parts $(z \geq 0)$ are in ${}_{\varphi}C_{\theta}$ and the lower parts $(z \leq 0)$ are in ${}_{\varphi}C_{\pi-\theta}$. In the same way, the intersection curves of $\mathbf{S}^2$ and the elliptic cylinder defined by

$$(2\sin\theta)y^2 + \sin\theta(1+\cos\varphi)z^2 + (2\cos\theta\sin\varphi)z = \sin\theta(1-\cos\varphi)$$

are also divided into two parts. This completes the proof. ■

In the second place, let us investigate a geometric property of ${}_{\varphi}C_{\varphi}$. Figure 2.4 shows the views from z-axis. We can see that the red curves orthogonally intersect at the north pole N.

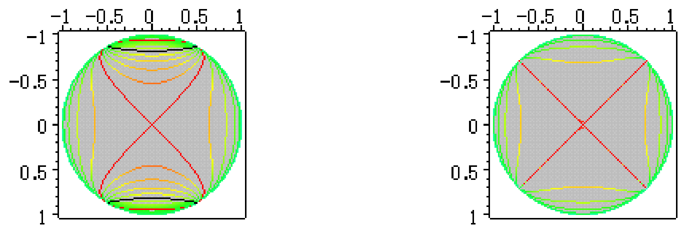

Figure 2.4 View of contours from z-axis($\varphi = \pi/3$ (left), $\varphi = \pi/2$ (right)).

Proposition 2.2 For any φ, the curve ${}_{\varphi}C_{\varphi}$ has singularities at the north and south poles with orthogonal tangent directions.

Proof. By the symmetry of $\mathbf{S}^2$, it is sufficient to prove the case of the north pole N. By Equation (2.1), the upper part of ${}_{\varphi}C_{\varphi}$ satisfies

$$(2\cos\varphi)z = 2y^2 + (1+\cos\varphi)z^2 - 1 + \cos\varphi. \tag{2.7}$$

If $\varphi = \pi/2$, the equation above is $2y^2 + z^2 = 1$, that is, $x^2 - y^2 = 0$. Hence ${}_{\varphi}C_{\varphi}$ is composed of two great circles which orthogonally intersect at N as in Figure 2.4(right).
If $\varphi \neq \pi/2$, squaring the both sides of Equation (2.7),

$$4\cos^2\varphi\,(1 - x^2 - y^2) = \{2\cos\varphi - (1+\cos\varphi)x^2 + (1-\cos\varphi)y^2\}^2,$$

equivalently,

$$4\cos\varphi(x^2 - y^2) + 2(\sin^2\varphi)x^2y^2 - (1+\cos\varphi)^2x^4 - (1-\cos\varphi)^2y^4 = 0.$$

In a neiborhood of N, x and y are sufficiently small, hence ignoring the higher order terms x^2y^2, x^4, and y^4 one also has $x^2 - y^2 = 0$. This completes the proof. ■

In the third place, let us investigate another geometric property of ${}_{\varphi}C_{\theta}$ around two points A and B. The next property is also true in Euclidean geometry.

Proposition 2.3 For any φ and θ, the angle between the curve ${}_{\varphi}C_{\theta}$ and the arc $\overline{AB}$ at A or B is $\pi-\theta$.

Proof. Let us calculate the angle at $A=(\cos(\varphi/2),-\sin(\varphi/2),0)$ under the case that $z\geq 0$. By changing coordinates such as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos(\varphi/2) & -\sin(\varphi/2) & 0 \\ \sin(\varphi/2) & \cos(\varphi/2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

the first one of Equations (2.1) becomes

$$\sin\theta\cos\varphi(y'^2+z'^2)-(\sin\theta\sin\varphi)x'y'-(\cos\theta\sin\varphi)z'=0 \quad (z'\geq 0).$$

In a neiborhood of $(x',y',z')=(1,0,0)$, y' and z' are sufficiently small and x' is nearly equal to 1, hence ignoring the higher order terms y'^2 and z'^2, and substituting $x'=1$, one has $z'=-(\tan\theta)y'$. This completes the proof. ■

Finally, let us investigate ${}_{\varphi}C_{\pi/2}$ (black curve in the left of Figure 2.3). The next proposition shows that ${}_{\varphi}C_{\pi/2}$ is a one-parameter family of spherical conic on $\mathbf{S}^2$ (see, [3]).

Proposition 2.4 For $\varphi\neq\pi/2$, ${}_{\varphi}C_{\pi/2}$ is a spherical conic (a pair of spherical ellipses), i.e.,

$${}_{\varphi}C_{\pi/2} = \{P|\overline{PF_1}+\overline{PF_2}=l\}\cup\{P|\overline{PF_1^*}+\overline{PF_2^*}=l\}$$

where $F_1=\left(\dfrac{2\sqrt{\cos\varphi}}{1+\cos\varphi},0,\dfrac{1-\cos\varphi}{1+\cos\varphi}\right)$, $F_2=\left(\dfrac{2\sqrt{\cos\varphi}}{1+\cos\varphi},0,-\dfrac{1-\cos\varphi}{1+\cos\varphi}\right)$, $l=2\sin^{-1}(\tan(\varphi/2))(>\varphi)$ and F_1^* (resp. F_2^*) is the antipodal point of F_1 (resp. F_2), respectively.

Proof. Let us show that $P=(x,y,z)$ such that $\overline{PF_1}+\overline{PF_2}=l$ satisfies Equation (2.6), i.e.,

$$2y^2+(1+\cos\varphi)z^2=1-\cos\varphi.$$

Let $c=\cos^{-1}\left(\dfrac{2\sqrt{\cos\varphi}}{1+\cos\varphi}\right)$ be the distance between F_1 and the mid point of A and B (the center of the ellipse). Then $F_1=(\cos c,0,\sin c)$ and $F_2=(\cos c,0,-\sin c)$, and note that

$$\cos^2 c=\frac{4\cos\varphi}{(1+\cos\varphi)^2} \quad \text{and} \quad \sin^2 c=\frac{(1-\cos\varphi)^2}{(1+\cos\varphi)^2}.$$

Since $\cos(\overline{PF_1}+\overline{PF_2})=\cos l$ and

$$\cos\overline{PF_1}=x\cos c+z\sin c, \quad \sin\overline{PF_1}=\sqrt{1-(x\cos c+z\sin c)^2},$$
$$\cos\overline{PF_2}=x\cos c-z\sin c, \quad \sin\overline{PF_2}=\sqrt{1-(x\cos c-z\sin c)^2},$$

one has

$$x^2\cos^2 c-z^2\sin^2 c-\cos l=\sqrt{(1-(x\cos c+z\sin c)^2)(1-(x\cos c-z\sin c)^2)}.$$

Squaring the both sides of the equation above,

$$\cos^2 l - 2\cos l(x^2\cos^2 c - z^2\sin^2 c) = 1 - 2(x^2\cos^2 c + z^2\sin^2 c),$$

or equivalently,

$$x^2\cos^2 c\sin^2\frac{l}{2} + z^2\sin^2 c\cos^2\frac{l}{2} = \sin^2\frac{l}{2}\cos^2\frac{l}{2}.$$

Using the equations $\sin^2\frac{l}{2} = \tan^2\frac{\varphi}{2} = \frac{1-\cos\varphi}{1+\cos\varphi}$ and $\cos^2\frac{l}{2} = \frac{2\cos\varphi}{1+\cos\varphi}$,

$$2x^2 + (1-\cos\varphi)z^2 = 1+\cos\varphi,$$

therefore,

$$2y^2 + (1+\cos\varphi)z^2 = 1-\cos\varphi.$$

In a similar way, it is showed that $P = (x, y, z)$ such that $\overline{PF_1^*} + \overline{PF_2^*} = l$ also satisfies the equation above. This completes the proof. ■

3. Stochastic property of visual angle

In this section, let us introduce a stochastic property of visual angle. For a given angle $\angle AOB$ in $\mathbf{R}^3$, take a random point P distributed uniformly on the unit sphere $\mathbf{S}^2$ centered at O. Then the visual size $\angle_P AOB$ is a random variable, which is called the *random visual size* of $\angle AOB$.

Theorem 3.1 For any angle $\angle AOB$, the expected value of the random visual size $\angle_P AOB$ is equal to the true size of $\angle AOB$, that is,

$$\mathbf{E}(\angle_P AOB) = \angle AOB.$$

Proof. Let $\angle AOB$ be an angle of size $\angle AOB$, and let P be a random point on the unit sphere $\mathbf{S}^2$ centered at O. We may suppose that A and B lie on $\mathbf{S}^2$. Then the spherical distance $\overline{AB}$ between A and B is equal to $\angle AOB$ and recall that $\angle_P AOB$ is equal to the interior angle $\angle P$ of the spherical triangle ΔAPB.

Let us assume that two points A and B are on the equator of $\mathbf{S}^2$. If it is proved that the expected value $\mathbf{E}(\angle_P AOB)$ restricted to any fixed latitude meridian is equal to $\angle AOB$, the proof of Theorem 1 has completed. Hence, in the rest of the proof, let us restrict the random point P to any fixed latitude meridian

$$L_\phi := \{\, P \in \mathbf{S}^2 \;\mid \angle NOP = \phi \,\},$$

where N is the north pole of $\mathbf{S}^2$.

First, let us prove the case of $\angle AOB = 2\pi/n$ where n is an integer greater than 1.

Divide the equator into n equal parts,

$$\overline{A_1A_2} = \overline{A_2A_3} = \cdots = \overline{A_{n-1}A_n} = \overline{A_nA_1} = 2\pi/n.$$

Then, for any point P,

$$\angle_P A_1OA_2 + \angle_P A_2OA_3 + \cdots + \angle_P A_{n-1}OA_n + \angle_P A_nOA_1 = 2\pi. \tag{3.1}$$

By the rotation with the axis ON and angle $2\pi/n$, the restricted expected value $\mathbf{E}|_{L_\phi}(\angle_P A_2OA_3)$ is equal to $\mathbf{E}|_{L_\phi}(\angle_P A_1OA_2)$, and so on. Therefore, taking the expectation of Equation (3.1), the linearity of expectation implies that

$$n\,\mathbf{E}|_{L_\phi}(\angle_P A_1 O A_2) = 2\pi. \tag{3.2}$$

Equation (3.2) shows that $\mathbf{E}|_{L_\phi}(\angle_P AOB) = \angle AOB$ in the case of $\angle AOB = 2\pi / n$.

In a similar way, we can prove that $\mathbf{E}|_{L_\phi}(\angle_P AOB) = \angle AOB$ in the case of $\angle AOB = q\pi$ where q is a rational number less than 1.

Finally, it is clear that the expected value $\mathbf{E}|_{L_\phi}(\angle_P AOB)$ is a continuous and monotone increasing function of the size of $\angle AOB$. Therefore, we can prove that $\mathbf{E}|_{L_\phi}(\angle_P AOB) = \angle AOB$ in the case of $\angle AOB = r\pi$ where r is a real number less than 1. We have completed the proof of Theorem 3.1.■

Theorem 3.1 indicates that when we observe an angle from several viewpoints, each chosen at random, the average visual size is approximately equal to the true size.

4. Conclusions

We see various angles in our daily lives, however, we are usually unconscious of their appearances. We started our discussion from the appearance, that is, the visual angle. The visual angle naturally introduces us some algebraic curves on the unit sphere. These curves have several interesting geometric properties. Furthermore, we have seen that the visual angle has a stochastic property. In [2] and [6], visual angles of several geometric shapes (rectangle, orthogonal axes and cube) are investigated. In addition, visual angle is extended to visual solid angle in the four dimensional Euclidean space $\mathbf{R}^4$ ([4]). In this way, visual angle gives us various interests in mathematics.

5. References

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